The Role of Quadrature in Meshfree Methods: Variational Consistency in Galerkin Weak Form and Collocation in Strong Form

by
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Quadrature Issues in Galerkin Meshfree Methods: Loss of Galerkin Orthogonality Leading to Suboptimal Convergence

Meshfree methods introduce new approximations for solving PDEs without using conventional meshing topology, as shown in *figure 1*. This reduces the strong tie between the quality of the discretization and the quality of the approximation, and can ultimately ease the difficulty of mesh distortion problems. The functions used in meshfree methods such as moving least-squares (MLS) and reproducing kernel (RK) approximations can be constructed with arbitrary order of completeness and continuity. These unique properties provide the opportunity to solve problems that are difficult or impossible to be solved by conventional finite element methods, and *it admits the development of paradigms for solving PDEs without being restricted to Galerkin type procedures.*

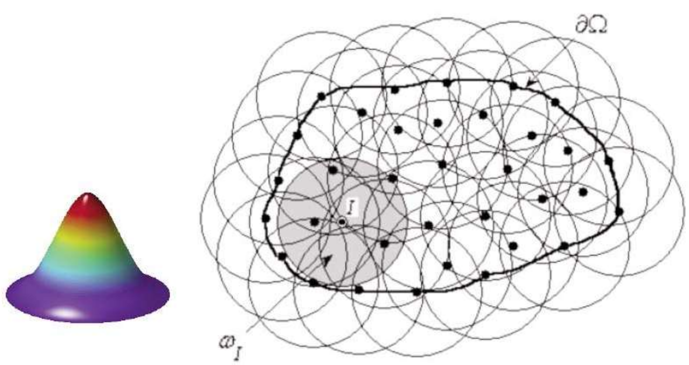


Figure 1: Meshfree approximation function and domain discretization by points

A critical issue in Galerkin meshfree methods is domain integration, typically carried out by Gauss integration (GI) or direct nodal integration (DNI). The MLS and RK approximations shown in *figure 1* are typically *rational functions with overlapping supports*. These approximations are capable of reproducing monomials of arbitrary order in arbitrary discretizations; however, this completeness property does not guarantee optimal rates of convergence in the Galerkin solution of PDEs if the domain integration is not sufficiently accurate. The quadrature error in the Galerkin equation also leads to the *loss of Galerkin orthogonality* as shown in *figure 2*, where u^h and u^{hh} are the solutions of the Galerkin equation without and with quadrature error, respectively. *Violation of Galerkin orthogonality leads to the loss of the best approximation property of the Galerkin solution* according to Strang's first Lemma [1]. This can be demonstrated by solving a PDE with a linear solution given in *figure 3*, where RK approximation with linear basis is introduced, and the exact linear solution is not obtained. Errors in this linear problem are also reflected in sub-optimal convergence when solving the problem with a higher order solution shown in *figure 4*, where optimal convergence rates (2 in the L^2 norm and 1 in the semi-H^1 norm) are not obtained, even with 5x5 GI.

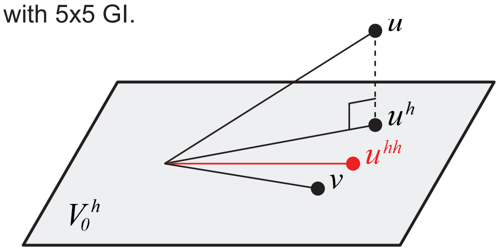
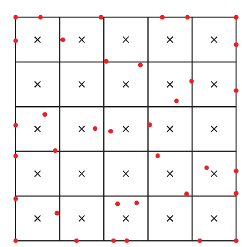


Figure 2: Projection of true solution u into finite dimensional space: Galerkin solution u^h without quadrature error and u^{hh} with quadrature error

$$\nabla^2 u = 0 \quad \text{in } \Omega$$
$$u = x + 2y \quad \text{on } \partial\Omega$$
$$\Omega = (-1, 1) \times (-1, 1)$$

MLS/RK: linear basis



L2 error norm

DNI	0.0321744
Gauss, 1x1	0.0068057
Gauss, 2x2	0.0016625
Gauss, 3x3	0.0005414
Gauss, 4x4	0.0002542
Gauss, 5x5	0.0000913

Figure 3: Linear exactness test of meshfree methods with linear basis using various domain integration methods (red solid dots: discretization points, cross symbols: Gauss quadrature points in GI)

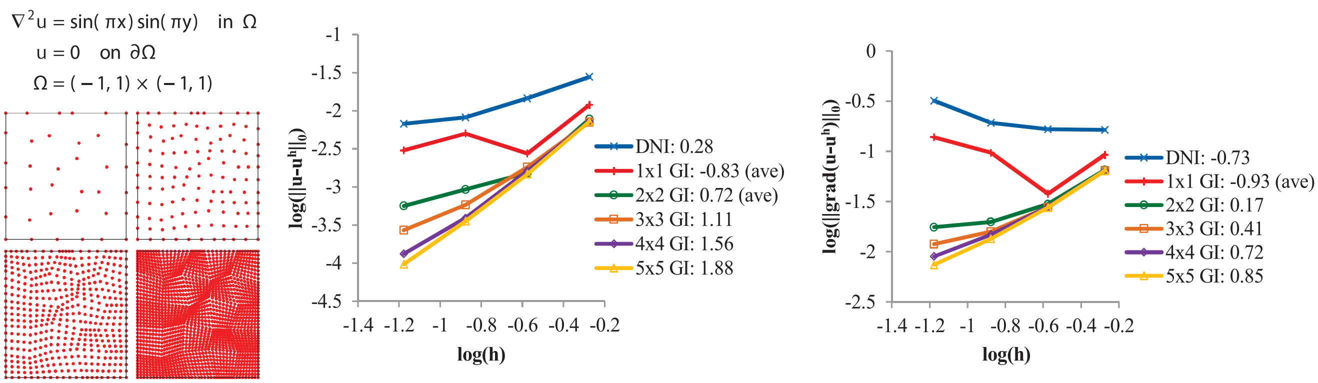


Figure 4: *Convergence of the Galerkin meshfree method with various integration methods for a Poisson problem with a higher order solution*

Recovery of Galerkin Orthogonality by Variational Consistency

One of the early attempts to recover Galerkin orthogonality in meshfree methods was the development of stabilized conforming nodal integration (SCNI) [2, 3], where the Galerkin orthogonality is approximated by imposing the following first order integration constraint to achieve *first order Galerkin exactness:*

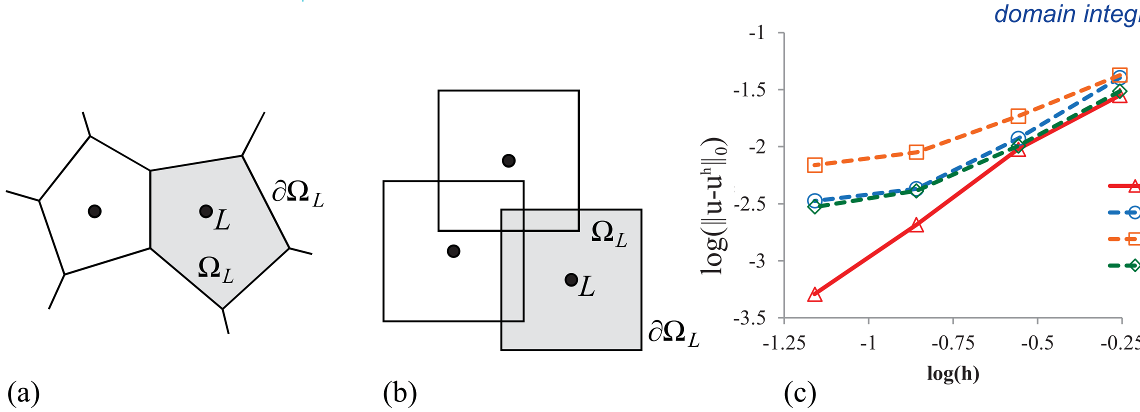
$$\hat{\int_{\Omega}} \nabla \Psi_I(\mathbf{x})\, d\Omega = \hat{\int_{\partial\Omega}} \Psi_I(\mathbf{x})\mathbf{n}\, d\Gamma \quad \forall I \tag{1}$$

Here, the superposed "^" denotes the numerical integration, $\Psi_I(\boldsymbol{x})$ is the meshfree approximation function associated with point I, and $\mathbf{n}$ is the unit outward surface normal. *A quadrature rule that satisfies the divergence condition in (1) recovers first order Galerkin orthogonality.* In SCNI [2], gradient smoothing under an assumed strain framework has been introduced to meet the integration constraint (1):

$$\bar{\nabla} u^h(\mathbf{x}_L) = \frac{1}{A_L}\int_{\Omega_L} \nabla u^h(\mathbf{x})\, d\Omega = \frac{1}{A_L}\int_{\partial\Omega_L} u^h(\mathbf{x})\mathbf{n}(\mathbf{x})\, d\Gamma \tag{2}$$

where $A_L = \int_{\Omega_L} d\Omega$ and Ω_L is the conforming representative domain of point L as shown in *figure 5(a)*. It can be shown that SCNI recovers quadratic rate of convergence in the L^2 norm when a linear basis is used as shown in *figure 5(c)*. When SCNI is simplified to stabilized nonconforming nodal integration (SNNI) [4] with nonconforming smoothing domains as shown in *figure 5(b)*, Galerkin orthogonality is lost and poor convergence is again encountered similar to that in GI and DNI as shown in *figure 5(c)*. *The recovery of Galerkin orthogonality in SCNI can be viewed as introducing a corrected finite dimensional space for the gradient as shown in figure 6, to recover the solutions u^h from u^{hh}.* The SCNI method has been enhanced with additional stabilization [5,6], and has also been applied to meshfree methods for plates and shells [7, 8, 9] and other meshfree methods such as the natural element method [10].

Figure 5: *(a) Conforming Voronoi smoothing domains for SCNI, (b) nonconforming smoothing domains for SNNI, (c) convergence in the L^2 norm for various domain integration methods*



A general framework for achieving n^{th} order Galerkin exactness (thus recovering n^{th} order Galerkin orthogonality) is the recent work in [11] which introduces the concept of *variational consistency (VC)*. VC dictates that for achieving n^{th} order Galerkin exactness, the following general n^{th} order integration constraints must be satisfied in addition to using n^{th} order completeness of trial functions:

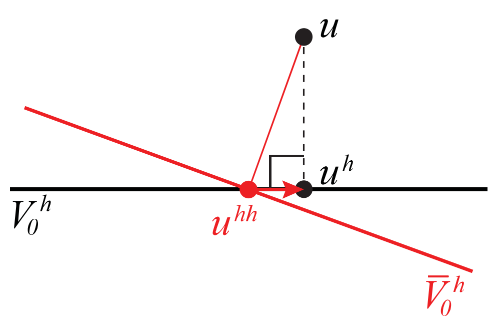
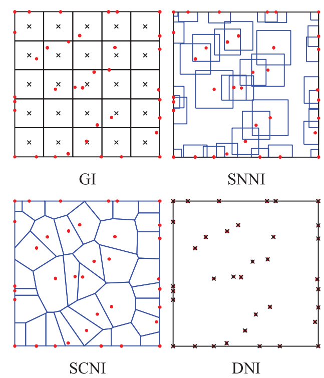


Figure 6: *Recovery of Galerkin orthogonality by a correction of finite dimensional space*

$$a\langle\hat{\Psi}_I,\mathbf{x}^\alpha\rangle_\Omega=-\langle\hat{\Psi}_I,L\mathbf{x}^\alpha\rangle_\Omega+\langle\hat{\Psi}_I,B\mathbf{x}^\alpha\rangle_{\partial\Omega}\;\forall I,\;\;|\alpha|=0,1,\cdots,n \qquad (3)$$

where $a\langle\cdot,\cdot\rangle_\Omega$ is the quadrature version of the bilinear form, $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ are the quadrature versions of inner products of two functions in the domain and on the boundary, respectively, L and B are the differential operator and the Neumann boundary operator of a boundary value problem, respectively, $\mathbf{x}^\alpha$ is the multi-dimensional representation of a monomial with degree α, $|\alpha|$ denotes the norm of multi-dimensional index, and $\hat{\Psi}_I$ is the test function. The equation in (3) states that a numerical integration by parts condition should hold for the inner product between test functions and the differential operator acting on n^{th} order monomials. *It can be shown that SCNI possesses first order variational consistency while GI, DNI, and SNNI are in general not variationally consistent with the meshfree approximation functions [11,12].*

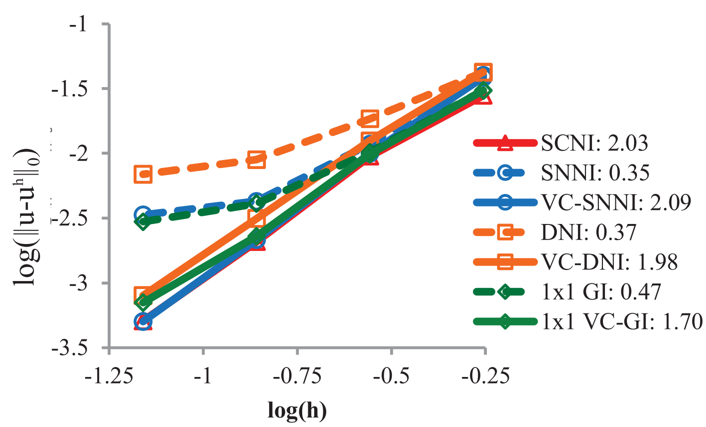
$\nabla^2 u = 0$ in Ω
$u = x + 2y$ on $\partial\Omega$
$\Omega = (-1,1)\times(-1,1)$



(a)

Method	L2 error
DNI	0.593025
SNNI	0.282710
SCNI	2.46E-14
1x1 GI	0.123775
2x2 GI	0.048582
3x3 GI	0.021067
4x4 GI	0.007910
5x5 GI	0.003436
VC-DNI	2.70E-14
VC-SNNI	1.67E-14
1x1 VC-GI	1.57E-14
2x2 VC-GI	2.71E-14
3x3 VC-GI	1.32E-14
4x4 VC-GI	1.77E-14
5x5 VC-GI	2.90E-14

(b)



(c)

Figure 7: *(a) Two dimensional linear exactness test, (b) L^2 errors in various domain integration methods and their VC corrected counterparts, (c) L^2 convergence rates*

Variational Consistency for Recovery of Optimal Convergence

VC provides a paradigm for formulating quadrature rules and test functions to achieve optimal convergence for a given PDE, and it can also be used to correct methods that are variationally inconsistent via modification of test functions to recover Galerkin orthogonality as shown in *figure 6*. The construction of the test function gradients proposed in [11] for achieving the n^{th} order integration constraint in (3) is performed by introducing a corrected test function gradient

$\nabla\hat{\Psi}_I=\nabla\Psi_I+\sum_{|\beta|\le n}\xi_{\beta I}\nabla\hat{\Psi}_I^\beta$, where $\{\Psi_I,\hat{\Psi}_I^\beta\}^n_{|\beta|=1}$ are linearly independent, and the coefficients $\xi_{\beta I}$ are solved from the following equation:

$$\sum_{|\beta|=1}^{n}A_{\alpha\beta I}\xi_{\beta I}=r_{\alpha I}\;\forall I,\;\;|\alpha|=0,1,\cdots,n \qquad (4)$$

where $A_{\alpha\beta I}$ and $r_{\alpha I}$ are the components of the linear system matrix and the residual vector, respectively, obtained from substituting

$\nabla\hat{\Psi}_I=\nabla\Psi_I+\sum_{|\beta|\le n}\xi_{\beta I}\nabla\hat{\Psi}_I^\beta$ into (3). The residual term $r_{\alpha I}$ represents the violation of integration constraint in (3). Here it is worth noting that with proper selection of $\nabla\hat{\Psi}_I^\beta$, $A_{\alpha\beta I}$ can be diagonalized for fast computation. Since the type of numerical integration to be corrected is arbitrary in (4), several variationally consistent methods have been derived under a unified framework [11].

(a)

(b)

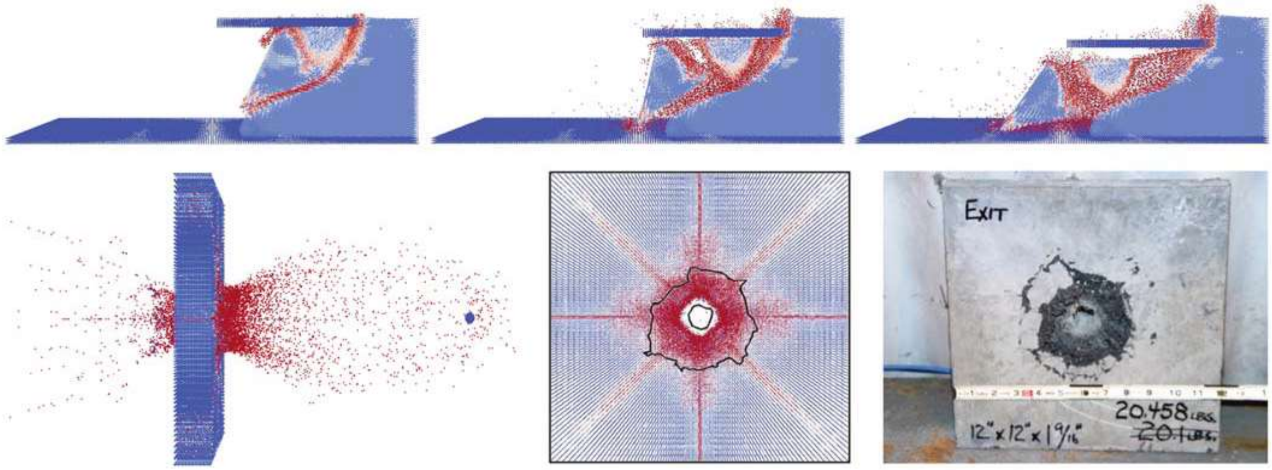


Figure 8:
Meshfree modeling of (a) slope instability, and (b) bullet penetration through a concrete plate

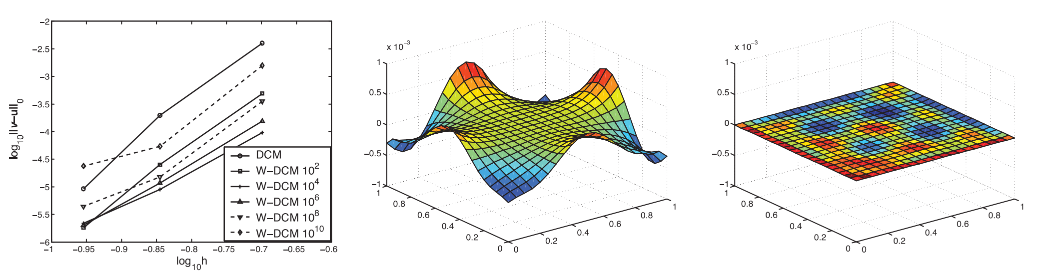
Using the integration schemes shown in *figure 7(a)*, a PDE with a linear solution shown in *figure 7(b)* is solved using the RK with linear basis. The results given in *figure 7(b)* demonstrate VC integration methods achieve the exact linear solution, and the results in *figure 7(c)* demonstrate the recovery of optimal convergence rates for domain integration methods corrected by VC for problems with higher order solution. VC recovery of higher order convergence for solving PDEs with higher order basis is also reported in [11]. *Figure 8* demonstrates the application of the Galerkin meshfree method to problems involving large deformations, failure and fragmentation processes [4, 13].

Strong Form Collocation Methods
Using the arbitrary smoothness of the meshfree approximation functions, collocation on the strong form has been proposed for meshfree methods, such as the finite point method [14], radial basis collocation methods (RBCM) [15, 16, 17, 18, 19, 20], and the reproducing kernel collocation method (RKCM) [21, 22].

These methods directly circumvent the quadrature issues associated with the Galerkin methods. From the standpoint of convergence, the compactly supported RK approximations with monomial reproducibility render algebraic convergence in RKCM, and the nonlocal radial basis functions (RBFs) with certain regularity offer exponential convergence in RBCM. However, two major dis-advantages have been identified in the strong form collocation method: (1) the unbalanced errors from the domain and boundary collocation equations lead to reduced accuracy and convergence rates, and (2) the linear system of RBCM is typically more ill-conditioned compared to those based on compactly supported approximations.

Figure 9:
(a) Convergence in the L^2 error norm (numbers associated with W-DCM denote the weights for the Dirichlet boundary collocation equations), (b) error distribution of DCM, (c) error distribution of W-DCM

$$\Delta u = (x^2 + y^2)e^{xy} \text{ in } \Omega$$
$$u = e^{xy} \text{ on } \partial\Omega$$
$$\Omega = (0,1) \times (0,1)$$



(a) (b) (c)

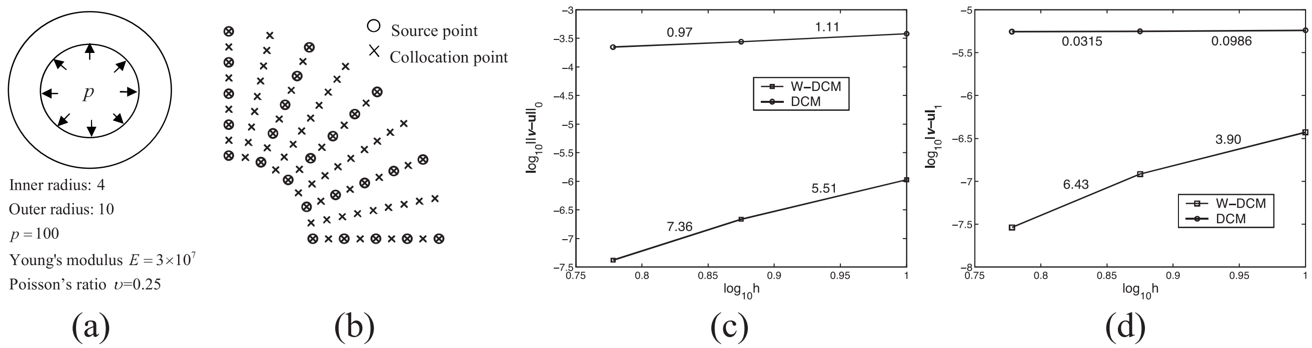


Figure 10:
(a) Problem statement of an infinite tube subjected to an internal pressure,
(b) discretization of the quarter model,
(c) convergence in the L^2 norm,
(d) convergence in the H^1 semi-norm

Convergence and Conditioning of Collocation Methods

The issue of unbalanced errors of the domain and boundary collocation equations in the strong form with direct collocation method (DCM) was first investigated in [20]. This work first showed that the *employment of least-squares functionals with quadrature rules constitutes an approximation of the strong form collocation method*. Based on error analysis of a least-squares functional, a strong form solution procedure using a weighted direct collocation method (W-DCM) was then proposed for enhanced accuracy and convergence [20]. The analysis given in [20] concludes that the *weight* $\sqrt{\alpha}$ *for the Dirichlet boundary collocation equations should be in the order of* $\alpha = (\kappa N_p)^2$ *for optimal convergence*, where κ is the maximum coefficient involved in the differential operator and the boundary operator, and N_p is the number of discrete points used in the discretization. Consider a Poisson problem shown in *figure 9* solved by the strong form collocation method using multiquadrics radial basis functions, where the domain is discretized by 6x6, 8x8, and 10x10 discrete points. It is seen in *figure 9(a)* that the proposed W-DCM with properly selected weights for the Dirichlet boundary on the order of $\alpha = (\kappa N_p)^2 \approx 10^4$ yields the best solution accuracy and convergence rate, and it is a significant enhancement over the DCM solution as seen in *figures 9(b) and 9(c)*. The superior performance of W-DCM over DCM can also be seen in the tube inflation problem shown in *figure 10*.

Standard RBF offers exponential convergence; however the RBCM method suffers from large condition numbers due to its "nonlocal" approximation. The work in [23] constructed a localized RBF using a partition of unity function, such as the RK function, as a localizing function to yield a local approximation with significantly enhanced conditioning while maintaining the exponential convergence of RBF.
In two-dimensional elasticity, the stability analysis in [23] shows that *the conditioning number of the RBCM of the order of* $O(h^{-8})$ *is reduced to* $O(h^{-3})$ *in the collocation method using RK localized RBF*. The RK localized RBF approach combined with the subdomain collocation method has been applied to problems with local features, such as problems with heterogeneity [24] and cracks [25]. It has also been shown that the collocation method with radial basis function (RBCM) can achieve very small dispersion error (<1%) compared to linear and quadratic finite elements [26]. Further, since the discrete system of the strong form collocation method is typically overdetermined and is solved by a least-squares method, mixed approximations can be introduced for constraint problems (such as incompressible problems) without suffering from instability due to violation of the LBB condition [27].

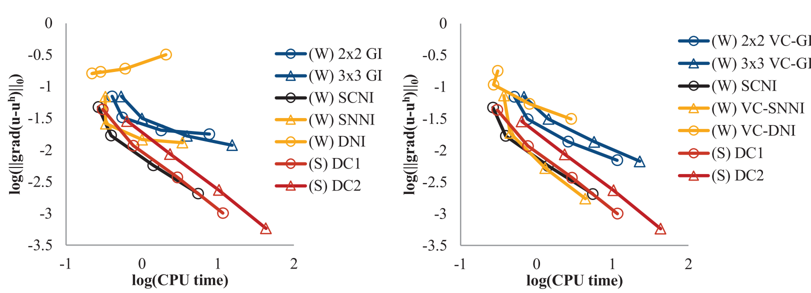


Figure 11:
Effectiveness comparison of meshfree methods based on weak and strong formulations

Comparison of Meshfree Methods based on Weak and Strong Formulations

The requirement of VC for integration in the weak formulation to recover Galerkin orthogonality originates from the introduction of integration by parts in the construction of the Galerkin weak equation. The strong form collocation method, on the other hand, does not invoke integration *by parts and is thus not subjected to the VC condition for optimal convergence.* A comparison of the effectiveness of meshfree methods based on the Galerkin weak form approach (denoted with (W)) and collocation in the strong form (denoted with (S)) is shown in *figure 11*. In this study, RK approximations with linear bases are employed for the Galerkin weak form approach, and quadratic bases are used for the collocated strong form approach as necessary for convergence [22]. For the strong form collocation approach, the discrete points used as collocation points is denoted as DC1, and background collocation with half the spacing of the nodes is denoted as DC2. The results for the H^1 error norm show that the weak formulation using VC-SNNI and SCNI, and strong formulation using DC1 are the most effective methods for obtaining an accurate solution. It is interesting and informative to note that similar approaches ((S) DC2 and (W) 2x2 GI, and (S) DC1 and (W) DNI) can give drastically different results; without VC integration Galerkin methods can hardly compare in terms of efficiency. However with VC, Galerkin methods compare well to strong form collocation, particularly SCNI and VC-SNNI. A comparison of meshfree methods formulated based on weak and strong formulations is summarized in *table 1*. ●

Table 1:
Advantages and disadvantages of meshfree methods based on weak and strong formulations

Feature	Meshfree with Galerkin Weak Form	Meshfree with Strong Form Collocation
Overall effectiveness	Standard Galerkin methods with integration such as GI or DNI are not effective unless a VC method or a VC corrected integration is used. Out of all the methods, SCNI and VC-SNNI are the most effective.	Methods are effective as long as the number of collocation points is not too large. In particular, DC1 and DC2 are most effective.
Integration/collocation points	Too few integration points lead to sub-optimal convergence for the standard Galerkin method. For VC integration methods, nodal integration methods are sufficient for convergence.	Maintains reasonable convergence even with few collocation points as long as boundary collocation equations are properly weighted.
Nonuniform point distributions	Leads to sub-optimal convergence in the standard Galerkin method without higher order quadrature or VC correction. VC corrected integration methods yield good convergence and the solution is insensitive to the nonuniformity of the nodal distribution.	Solution accuracy and convergence is more sensitive to nonuniform point distributions than the Galerkin method.
Implementation	Similar to FEM.	Easier than FEM, except for nonlinear problems.
Additional advantages /disadvantages	• Easy coupling with FEM and has similar coding structure. • Compactly supported approximation functions used in weak form yields well-conditioned discrete system. • Very effective for large deformations, fracture problems, and h-adaptive refinements.	• RBCM yields exponential convergence and nearly zero dispersion error, but the discrete systems have high condition numbers. • RKCM is well-conditioned, but requires at least 2nd order basis in the RK functions. • No LBB instability in the mixed formulation and thus is attractive for constrained problems. • Requires 3rd derivatives in linearized equation of nonlinear 2nd order PDE.

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